

## CONFORMALLY NATURAL EXTENSIONS REVISITED

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ABSTRACT. In this note we revisit the notion of conformal barycenter of a measure on  $\mathbb{S}^n$  as defined by Douady and Earle [D-E]. The aim is to extend rational maps from the Riemann sphere  $\widehat{\mathbb{C}} \approx \mathbb{S}^2$  to the (hyperbolic) three ball  $\mathbb{B}^3$  and thus to  $\mathbb{S}^3$  by reflection. The construction which was pioneered by Douady and Earle in the case of homeomorphisms actually gives extensions for more general maps such as entire transcendental maps on  $\mathbb{C} \subset \widehat{\mathbb{C}}$ . And it works in any dimension.

## 1. INTRODUCTION

Let  $G = G_n$  denote the group of Möbius transformations of Möbius space  $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$  preserving the  $n$ -sphere  $\mathbb{S}^n$ :

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

as well as the enclosed ball  $\mathbb{B}^{n+1}$ . Then each element  $g \in G$  acts on  $\mathbb{B}^{n+1}$  as a hyperbolic isometry, that is it preserves the Riemannian metric  $2|d\mathbf{x}|/(1 - |\mathbf{x}|^2)$ . Moreover each  $g$  is conformal and thus also acts as a conformal automorphism of both  $\mathbb{S}^n$  and  $\mathbb{B}^{n+1}$ . Mostow [Mo] proved that any conformal isomorphism of  $\mathbb{B}^{n+1}$  and/or  $\mathbb{S}^n$  is an element of  $G$ , so that we may also define  $G$  as the conformal automorphism group of  $\mathbb{B}^{n+1}$  and/or  $\mathbb{S}^n$ . We let  $G_+ = G_{n,+}$  denote the index two subgroup consisting of orientation preserving conformal automorphisms. And we let  $c$  denote the reflection in the coordinate plane  $x_{n+1} = 0$ , so that  $G$  is generated by  $G_+$  and  $c$ , i.e.  $G = \langle G_+, c \rangle$ .

We equip  $\mathbb{S}^n$  with the Spherical metric, which is the infinitesimal metric induced by the Euclidean metric on the ambient space  $\mathbb{R}^{n+1}$ . And we denote by  $R = R_n$  the subgroup consisting of Euclidean isometries, and by  $R_+ := G_+ \cap R$  the subgroup of orientation preserving rigid rotations. Then  $R$  is also the stabilizer of the origin  $\mathbf{0}$ . For  $\mathbf{w} \in \mathbb{B}^{n+1}$  define  $g_{\mathbf{w}} \in G_+$  by

$$(1) \quad g_{\mathbf{w}}(\mathbf{x}) = \frac{\mathbf{x}(1 - |\mathbf{w}|^2) + \mathbf{w}(1 + |\mathbf{x}|^2 + 2\langle \mathbf{w}, \mathbf{x} \rangle)}{1 + |\mathbf{w}|^2|\mathbf{x}|^2 + 2\langle \mathbf{w}, \mathbf{x} \rangle},$$

where  $\langle \cdot, \cdot \rangle$  denotes the Euclidean inner product. Then  $g_{\mathbf{w}}$  preserves the line segment  $[-\mathbf{w}/|\mathbf{w}|, \mathbf{w}/|\mathbf{w}|]$ , fixes  $\pm \mathbf{w}/|\mathbf{w}|$  and  $g_{\mathbf{w}}^{-1} = g_{-\mathbf{w}}$ . Moreover for  $0 < r < 1$  let  $\mathbf{r} = (r, 0, \dots, 0) = r\mathbf{e}_1$  and write  $g_r := g_{\mathbf{r}}$ , where in general  $\mathbf{e}_j$  denotes the  $j$ -th element of the standard orthonormal basis for  $\mathbb{R}^{n+1}$ . Any  $g_{\mathbf{w}}$  can be written  $g_{\mathbf{w}} = \rho \circ g_r$ , where  $r = |\mathbf{w}|$  and  $\rho = g_{\mathbf{w}} \circ g_r^{-1} \in R_+$ . Moreover any element  $g \in G_+$  can be written in a unique way as

$$g = g_{\mathbf{w}} \circ \rho',$$

where  $\mathbf{w} = g(\mathbf{0})$ , and

$$\rho' = g_{\mathbf{w}}^{-1} \circ g = g_{-\mathbf{w}} \circ g \in R_+.$$

So that in fact  $G_+ = \langle R_+, (g_r)_{0 < r < 1} \rangle$  and  $G = \langle R_+, (g_r)_{0 < r < 1}, c \rangle$ .

We can identify  $\widehat{\mathbb{R}}^n$  with  $\mathbb{S}^n$  via stereographic projection of the central plane  $x_{n+1} = 0$  in  $\widehat{\mathbb{R}}^{n+1}$  or equivalently through reflection in the sphere  $\mathbb{S}^n(\mathbf{e}_{n+1}, \sqrt{2})$ . In the case  $n = 2$  stereographic projection identifies  $\overline{\mathbb{C}} = \mathbb{CP}^1$  with  $\mathbb{S}^2$ . And in the  $\mathbb{C}$  coordinate an orientation preserving Möbius-transformation  $g$  preserving the unit circle can be written  $g = g_w(\rho z)$ , where  $|\rho| = 1$  and

$$g_w(z) = \frac{z + w}{1 + \overline{w}z}, \quad \text{where } w \in \mathbb{D}.$$

Following Douady and Earle the group  $G$  operates on  $\mathbb{B}^{n+1}$ ,  $\partial\mathbb{B}^{n+1} = \mathbb{S}^n$ , on the set of probability measures  $\mathcal{P}(\mathbb{S}^n)$  and on the vector space  $\mathcal{F}(\mathbb{B}^{n+1})$  of continuous vector fields on  $\mathbb{B}^{n+1}$ . That is

$$\begin{aligned} g \cdot \mathbf{z} &= g(\mathbf{z}), & \text{for } \mathbf{z} \in \overline{\mathbb{B}^{n+1}}, \\ (g \cdot \mu)(A) &= g_*\mu(A) = \mu(g^{-1}(A)), & \text{for } \mu \in \mathcal{P}(\mathbb{S}^n) \text{ and } A \subset \mathbb{S}^n \text{ a Borel subset,} \\ (g \cdot \mathbf{v})(g(\mathbf{z})) &= g_*(\mathbf{v})(g(\mathbf{z})) = D_{\mathbf{z}}g(\mathbf{v}(\mathbf{z})), & \text{for } \mathbf{v} \in \mathcal{F}(\mathbb{B}^{n+1}) \text{ and } \mathbf{z} \in \mathbb{B}^{n+1}. \end{aligned}$$

Here  $D_{\mathbf{z}}g$  denotes the differential of  $g$  at  $\mathbf{z}$ . The group  $G \times G$  operates on the spaces  $\text{End}(\mathbb{B}^{n+1}), \mathcal{C}(\mathbb{B}^{n+1})$  and  $\text{End}(\mathbb{S}^n), \mathcal{C}(\mathbb{S}^n)$  of endomorphisms and continuous endomorphisms of  $\mathbb{B}^{n+1}$  and  $\mathbb{S}^n$  respectively by

$$(g, h)\phi := g \circ \phi \circ h^{-1}.$$

If  $G$  operates on the spaces  $\mathbb{X}$  and  $\mathbb{Y}$  then a map  $T : \mathbb{X} \rightarrow \mathbb{Y}$  is called  $G$  equivariant or conformally natural if

$$\forall g \in G, \quad \forall x \in \mathbb{X} \quad : T(g \cdot x) = g \cdot T(x).$$

And if  $G \times G$  operates on both  $\mathbb{X}$  and  $\mathbb{Y}$  then conformal naturality of  $T$  is taken to mean  $G \times G$ -equivariance.

Douady and Earle introduced the idea of Conformal Barycenter for probability measures on  $\mathbb{S}^1 \subset \mathbb{C}$  and more generally on  $\mathbb{S}^n \subset \mathbb{R}^n$ . They used the conformal barycenter to define conformally natural extensions of self-homeomorphisms of  $\mathbb{S}^n$ . We shall in this note study the application of the Douady-Earle extension operator to a much wider class of maps than just homeomorphisms. More precisely let  $\eta_{\mathbf{0}}$  denote the normalized standard Euclidean or Lebesgue probability measure on  $\mathbb{S}^n$ . And let  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be a measureable endomorphism, for which the push-forward  $f_*(\eta_{\mathbf{0}})$  of  $\eta_{\mathbf{0}}$  by  $f$  is absolutely continuous with respect to  $\eta_{\mathbf{0}}$ . We shall show that the Douady-Earle extension operator also yields a conformally natural extensions of maps such as  $f$ , to non-constant self maps also denoted by  $f$ ,  $f : \overline{\mathbb{B}_{n+1}} \rightarrow \overline{\mathbb{B}_{n+1}}$ , which are real-analytic in the interior and continuous, whenever the original map  $f$  is continuous. In particular we obtain extensions of rational and entire transcendental maps of  $\mathbb{S}^2 \approx \mathbb{CP}^1$  to the hyperbolic three-space  $\mathbb{B}^3$ . And of course by reflection and renewed stereographic projection to an endomorphism of  $\mathbb{S}^3$ .

The motivation for this note comes from talks by Bill Thurston at a workshop in Roskilde 2010, where he asked: "What is the three-manifold of a rational map?" or

"How can we in a natural way extend rational maps to  $\mathbb{B}^3$ ?" The answer we propose to the second question is: Use the Douady-Earle extension.

In order to be self-contained we shall start by reviewing the Douady-Earle construction of the conformal barycenter and the Douady-Earle extension in any dimension.

## 2. CONFORMAL BARYCENTERS

**2.1. Harmonic measure.** Denote by  $\eta_0$  the normalized Euclidean Lebesgue measure on  $\mathbb{S}^n$ ,

$$\eta_0(A) = \frac{1}{\text{Vol}(\mathbb{S}^n)} \int \dots \int_A L, \quad \text{Vol}(\mathbb{S}^n) = \int \dots \int_A L,$$

where  $L$  denotes Lebesgue measure. We shall henceforth also write

$$\eta_0(A) = \int_A d\eta_0.$$

Then  $\eta_0$  is invariant under  $R$ , i.e.  $g_*(\eta_0) = \eta_0$  for every element  $g \in R$ .

For  $\mathbf{w} \in \mathbb{B}^{n+1}$  the harmonic measure with center  $\mathbf{w}$  is the measure  $\eta_{\mathbf{w}} = (g_{\mathbf{w}})_*(\eta_0)$ . Note that by the above  $\eta_{\mathbf{w}} = g_*(\eta_0)$  for any  $g \in G$  with  $g(0) = \mathbf{w}$ .

Also note that since each  $g \in G$  is conformal

$$|\text{Jac}_g(\mathbf{z})| = \|\text{Jac}_g(\mathbf{z})\|^n,$$

where  $\|\cdot\|$  denotes the operator norm and  $|\cdot|$  denotes determinant. In the 2-dimensional and thus 1-complex dimensional case one computes for  $g_w$  and  $|z| = 1$ :

$$|g'_w(z)| = \frac{1 - |w|^2}{|z + w|^2}.$$

Thus in real dimension  $n$  we obtain for  $\mathbf{z} \in \mathbb{S}^n$  and  $\mathbf{w} \in \mathbb{B}_{n+1}$ :

$$|\text{Jac}_{g_{\mathbf{w}}}(\mathbf{z})| = \left( \frac{1 - |\mathbf{w}|^2}{|\mathbf{z} + \mathbf{w}|^2} \right)^n$$

and hence by the change of variables formula

$$\eta_{\mathbf{w}}(A) = \int_A 1 d\eta_{\mathbf{w}} = \int_A 1 (g_{\mathbf{w}})_*\eta_0 = \int_{g_{-\mathbf{w}}(A)} 1 d\eta_0 = \int_A \left( \frac{1 - |\mathbf{w}|^2}{|\mathbf{z} - \mathbf{w}|^2} \right)^n d\eta_0.$$

**The Conformal Barycenter of a measure.** Let us define a probability measure to be *admissible*, if it has no atoms of mass greater than or equal to  $1/2$ . And let  $\mathcal{P}'(\mathbb{S}^n)$  denote the space of admissible probability measures. To each admissible probability measure  $\mu \in \mathcal{P}'(\mathbb{S}^n)$  we shall assign a point  $B(\mu) \in \mathbb{B}^{n+1}$  so that the map  $\mu \mapsto B(\mu) : \mathcal{P}'(\mathbb{S}^n) \rightarrow \mathbb{B}^{n+1}$  is conformally natural and normalized by

$$(2) \quad B(\mu) = \mathbf{0} \quad \Leftrightarrow \quad \int_{\mathbb{S}^n} \underline{\zeta} d\mu(\underline{\zeta}) = \mathbf{0}$$

**Proposition 1.** *The mapping  $V : \mathcal{P}(\mathbb{S}^n) \rightarrow \mathcal{F}(\mathbb{B}^{n+1})$ , which to a probability measure  $\mu \in \mathcal{P}'(\mathbb{S}^n)$  assigns the vector field*

$$(3) \quad V_{\mu}(\mathbf{w}) = \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\underline{\zeta}) d\mu(\underline{\zeta}), \quad \mathbf{w} \in \mathbb{B}^{n+1}$$

is the unique conformally natural such map satisfying the normalizing condition

$$(4) \quad V_\mu(\mathbf{0}) = \frac{1}{2} \int_{\mathbb{S}^n} \underline{\zeta} \, d\mu(\underline{\zeta}).$$

The normalizing factor  $\frac{1}{2}$  is inessential, but kept here in order to make  $V_\mu$  asymptotically a hyperbolic unit vector field at  $\infty$ , when  $\mu$  has no atoms.

*Proof.* Equivariance or conformal invariance is equivalent to

$$\forall g \in G, \forall \mathbf{w} \in \mathbb{B}^{n+1} : \quad V_{g_*\mu}(g(\mathbf{w})) = (g \cdot V_\mu)(g(\mathbf{w})) = D_{\mathbf{w}}g(V_\mu(\mathbf{w}))$$

Thus the normalizing condition (4) is invariant under the subgroup  $R$  stabilizing the origin, because such maps are linear. And for  $g = g_{-\mathbf{w}} = g_{\mathbf{w}}^{-1}$  with  $g_{-\mathbf{w}}(\mathbf{w}) = \mathbf{0}$ , the above formula implies

$$\forall \mathbf{w} \in \mathbb{B}^{n+1} : \quad V_\mu(\mathbf{w}) = D_{\mathbf{0}}g_{\mathbf{w}}V_{(g_{-\mathbf{w}})_*\mu}(\mathbf{0})$$

Thus the mapping  $\mu \mapsto V_\mu$  is conformally natural if and only if

$$\begin{aligned} V_\mu(\mathbf{w}) &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} \underline{\zeta} ((g_{-\mathbf{w}})_*\mu)(\underline{\zeta}) \\ &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\underline{\zeta}) \, d\mu(\underline{\zeta}) \\ (5) \quad &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} \frac{\underline{\zeta}(1 - |\mathbf{w}|^2) - 2\mathbf{w}(1 - \langle \underline{\zeta}, \mathbf{w} \rangle)}{1 + |\mathbf{w}|^2 - 2\langle \underline{\zeta}, \mathbf{w} \rangle} \, d\mu(\underline{\zeta}). \end{aligned}$$

□

Next we want to prove that

**Proposition 2.** *For each admissible measure  $\mu \in \mathcal{P}'(\mathbb{S}^n)$  the vector field  $V_\mu$  has a unique zero in  $\mathbb{B}^{n+1}$ .*

For the proof we shall use a few elementary lemmas, which are generalizations to dimension 3 and higher of the corresponding statements for the complex plane, as can be found in [D-E, Sections 2 and 11].

**Lemma 3.** *For any admissible probability measure  $\mu \in \mathcal{P}'(\mathbb{S}^n)$  any zero  $\mathbf{v} \in \mathbb{B}^{n+1}$  of the vector field  $V = V_\mu$  is an isolated stable equilibrium.*

*Proof.* By conformal naturality it suffices to consider the case  $\mathbf{v} = \mathbf{0}$ . Expanding the above formula (5) for  $V_\mu(\mathbf{w})$  to first order in  $\mathbf{w}$  we obtain:

$$\begin{aligned} V_\mu(\mathbf{w}) &= \frac{1}{2} \int_{\mathbb{S}^n} \underline{\zeta} - 2(\mathbf{w} - \underline{\zeta} \langle \mathbf{w}, \underline{\zeta} \rangle) \, d\mu(\underline{\zeta}) + o(|\mathbf{w}|) \\ &= V_\mu(\mathbf{0}) - \int_{\mathbb{S}^n} (\mathbf{w} - \underline{\zeta} \langle \mathbf{w}, \underline{\zeta} \rangle) \, d\mu(\underline{\zeta}) + o(|\mathbf{w}|) \\ &= - \int_{\mathbb{S}^n} (\mathbf{w} - \underline{\zeta} \langle \mathbf{w}, \underline{\zeta} \rangle) \, d\mu(\underline{\zeta}) + o(|\mathbf{w}|) \end{aligned}$$

since  $V_\mu(\mathbf{0}) = \mathbf{0}$ . Hence the Jacobian of  $V$  at  $\mathbf{v} = \mathbf{0}$  is given by

$$(6) \quad \text{Jac}_V(\mathbf{0})(\underline{\epsilon}) = - \int_{\mathbb{S}^n} (\underline{\epsilon} - \underline{\zeta} < \underline{\epsilon}, \underline{\zeta} >) d\mu(\underline{\zeta})$$

and thus  $\text{Jac}_V(\mathbf{0})$  is non singular. In fact  $\mathbf{v}$  is a sink since

$$(7) \quad < \underline{\epsilon}, \text{Jac}_V(\mathbf{0})(\underline{\epsilon}) > = - \int_{\mathbb{S}^n} (< \underline{\epsilon}, \underline{\epsilon} > - < \underline{\zeta}, \underline{\epsilon} > < \underline{\epsilon}, \underline{\zeta} >) d\mu(\underline{\zeta}) < 0.$$

□

Douady and Earle showed that if  $\mu(D_{\mathbb{S}^n}(\mathbf{e}_1, \pi/4)) \geq \frac{2}{3}$  then

$$(8) \quad < V_\mu(0), \mathbf{e}_1 > > 0,$$

where  $D_{\mathbb{S}^n}(\mathbf{e}_1, \delta)$  denotes the closed ball in  $\mathbb{S}^n$  of center  $\mathbf{e}_1$  and spherical radius  $\delta$ . This is sufficient to prove Proposition 2, if  $\mu$  has no atoms of mass  $\frac{1}{3}$  or higher. To prove the Proposition also, when no atom has mass  $\frac{1}{2}$  or higher, we need the following slight refinement:

**Lemma 4.** *Let  $\delta \in ]0, \sqrt{2}[$  and suppose  $\mu(D_{\mathbb{S}^n}(\mathbf{e}_1, \delta)) \geq (1 + \frac{\delta^2}{2})/2$ . Then*

$$< V_\mu(0), \mathbf{e}_1 > > 0.$$

*Proof.*

$$\begin{aligned} < V_\mu(0), \mathbf{e}_1 > &= \int_{D_{\mathbb{S}^n}(\mathbf{e}_1, \delta)} < \underline{\zeta}, \mathbf{e}_1 > d\mu(\underline{\zeta}) + \int_{\mathbb{S}^n \setminus D_{\mathbb{S}^n}(\mathbf{e}_1, \delta)} < \underline{\zeta}, \mathbf{e}_1 > d\mu(\underline{\zeta}) \\ &\geq (1 - \frac{\delta^2}{2})(1 + \frac{\delta^2}{2})/2 - 1 \cdot (1 - \frac{\delta^2}{2})/2 = \frac{\delta^2}{4}(1 - \frac{\delta^2}{2}) > 0. \end{aligned}$$

□

**Lemma 5.** *Suppose that  $\mu$  is admissible. Then there exists  $r \in ]0, 1[$  such that  $V_\mu(\mathbf{w})$  points inwards at any point  $\mathbf{w} \in \mathbb{B}^{n+1}$  with  $r \leq |\mathbf{w}| < 1$ , i.e.  $< V_\mu(\mathbf{w}), \mathbf{w} > < 0$ .*

*Proof.* Choose  $\delta \in ]0, \sqrt{2}[$  such that for any  $\underline{\zeta} \in \mathbb{S}^n$  :  $\mu(\{\underline{\zeta}\}) < (1 - \frac{\delta^2}{2})/2$ . Then there exists  $\epsilon \in ]0, \pi[$  such that for  $\underline{\zeta} \in \mathbb{S}^n$  :  $\mu(D_{\mathbb{S}^n}(\underline{\zeta}, \epsilon)) \leq (1 - \frac{\delta^2}{2})/2$ . Choose  $r \in ]0, 1[$  such that  $\forall \mathbf{w} \in \mathbb{B}^{n+1}$  with  $r \leq |\mathbf{w}| < 1$ :

$$\eta_{\mathbf{w}}(\mathbb{S}^n \setminus D_{\mathbb{S}^n}(\frac{\mathbf{w}}{|\mathbf{w}|}, \epsilon)) \leq \eta_{\mathbf{0}}(D_{\mathbb{S}^n}(\mathbf{e}_1, \delta))$$

Then it follows from Lemma 4 that  $V_\mu(\mathbf{w})$  points into the sphere  $S = |\mathbf{w}|\mathbb{S}^n$ : Let  $g \in G_+$  be any Möbius transformation mapping  $\mathbf{w}$  to  $\mathbf{0}$  and  $-\mathbf{w}/|\mathbf{w}|$  to  $\mathbf{e}_1$ . Then  $g(S)$  is a sphere through  $\mathbf{0}$  and with  $\mathbf{e}_1$  as an inwards pointing normal vector at  $\mathbf{0}$ . Moreover let  $\nu = g_*\mu$  then by conformal naturality  $g_*(V_\mu(\mathbf{w})) = V_\nu(0)$  and  $\nu$  satisfies the hypotheses of Lemma 4. □

*Proof.* of Proposition 2 Let  $\mu \in \mathcal{P}(\mathbb{S}^n)$  be any admissible measure, i.e. with no atom of mass  $1/2$  or higher. In Lemma 3 we have shown that any zero of the vector field  $V_\mu$  is an isolated stable equilibrium, i.e. the vector field points inwards on small spheres around the zero. Moreover by Lemma 5 the vector field  $V_\mu$  is pointing inwards near the boundary  $\mathbb{S}^n$  of  $\mathbb{B}^{n+1}$ . Hence by the Poincaré-Hopf theorem [Mi, see also Lemma 3, p 36]  $V_\mu$  has a unique zero  $B(\mu) \in \mathbb{B}^{n+1}$ . □

**Definition 6.** Define a conformally natural mapping  $B : \mathcal{P}'(\mathbb{S}^n) \longrightarrow \mathbb{B}^{n+1}$  by setting  $B(\mu)$  equal to the unique zero  $\mathbf{w} \in \mathbb{B}^{n+1}$  of the vector field  $V_\mu$ . Then  $B$  satisfies (2).

### 3. EXTENDING CONTINUOUS ENDOMORPHISMS OF $\mathbb{S}^n$ .

Let  $\mathcal{E}(\mathbb{S}^n)$  denote the space of endomorphisms  $\phi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$  such that  $\phi_*\eta_0$  has no atoms. For such mappings the measures  $\phi_*\eta_{\mathbf{z}}$  has no atoms neither for any  $\mathbf{z} \in \mathbb{B}^{n+1}$ . And let  $\text{End}(\overline{\mathbb{B}^{n+1}})$  denote the space of endomorphisms of  $\overline{\mathbb{B}^{n+1}}$ , whose restrictions to  $\mathbb{B}^{n+1}$  are endomorphisms of  $\mathbb{B}^{n+1}$ .

The Douady-Earle extension operator  $E$ , in the following denoted the D-E extension, which Douady and Earle studied for homeomorphisms is the map  $E : \mathcal{E}(\mathbb{S}^n) \longrightarrow \text{End}(\overline{\mathbb{B}^{n+1}})$  defined as follows: For  $\phi \in \mathcal{E}(\mathbb{S}^n)$  the mapping  $E(\phi) = \Phi : \overline{\mathbb{B}^{n+1}} \longrightarrow \overline{\mathbb{B}^{n+1}}$  is given by the formulas

$$(9) \quad \Phi(\mathbf{z}) = \begin{cases} \phi(\mathbf{z}), & \mathbf{z} \in \mathbb{S}^n, \\ B((\phi \circ g_{\mathbf{z}})_*(\eta_0)) = B(\phi_*(\eta_{\mathbf{z}})), & \mathbf{z} \in \mathbb{B}^{n+1} \end{cases}$$

Clearly the mapping  $\phi \mapsto E(\phi) = \Phi$  is conformally natural, i.e. for all  $g, h \in G$ :

$$E(g \circ \phi \circ h) = g \circ E(\phi) \circ h.$$

Moreover for any conformal automorphism  $g \in G$  we have  $E(g|_{\mathbb{S}^n}) = g$ , by conformal naturality of  $E$  and the fact that  $B(\eta_0) = \mathbf{0}$ . We can also formulate this as saying that the D-E extension operator extends the Poincaré extension operator. For  $n = 1$  at least we have a much stronger property: For inner functions, [R, Def. 17.14] that is for holomorphic selfmaps  $f : \mathbb{D} \longrightarrow \mathbb{D}$  of the unit disc  $\mathbb{D} \subset \mathbb{C}$  with boundary values in  $\mathbb{S}^1$  a.e., the D-E extension simply recovers  $f$  from its boundary values. More precisely it is well known that for bounded holomorphic functions (see [R, Th. 11.21]) the radial limit

$$f^\#(\zeta) = \lim_{r \nearrow 1} f(r\zeta)$$

exists for a.e.  $\zeta \in \mathbb{S}^1$  and satisfies the Cauchy formula:

$$(10) \quad \forall z \in \mathbb{D} : \quad f(z) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f^\#(\zeta)}{\zeta - z} d\zeta.$$

Moreover the space of such functions  $f^\#$  is the space of bounded measurable functions, whose negative Fourier coefficients are all equal to zero. For inner functions where  $|f^\#(\zeta)| = 1$  a.e. the measure  $f^\#_*\eta_0$  is absolutely continuous with respect to  $\eta_0$  (see [R, Th. 17.13]) and hence  $f^\#_*\eta_z$  is absolutely continuous with respect to  $\eta_0$  for any  $z \in \mathbb{D}$ , so that  $f^\# \in \mathcal{E}(\mathbb{S}^1)$ .

**Proposition 7.** If  $f : \mathbb{D} \longrightarrow \mathbb{D}$  is an inner function then

$$E(f^\#)(z) = f(z), \quad \forall z \in \mathbb{D}.$$

*Proof.* Let  $f$  be an arbitrary inner function and let  $z \in \mathbb{D}$  be arbitrary. We need to show that  $f(z) = E(f^\#)(z)$ . By conformal naturality we can suppose  $z = f(z) = 0$  as

we may precompose by  $g_z$  and postcompose by  $g_{-f(z)}$ . That is it suffices to prove that  $B(f_*\eta_0) = 0$  for any inner function  $f$  with  $f(0) = 0$ . We compute

$$2V_{f_*\eta_0}(0) = \int_{\mathbb{S}^1} \zeta(f_*\eta_0)(\zeta) = \int_{\mathbb{S}^1} f^\#(\zeta) d\eta_0(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f^\#(\zeta)}{\zeta} d\zeta = f(0) = 0.$$

□

**Lemma 8.** *Let  $\phi \in \mathcal{E}(\mathbb{S}^n)$  and let  $\Phi = E(\phi)$ . If  $\phi$  is continuous at some point  $\underline{\zeta}_0 \in \mathbb{S}^n$  then so is  $\Phi$ . In particular if  $\phi$  is continuous then  $\Phi$  is continuous on  $\mathbb{S}^n$ .*

We shall see in the next lemma that  $\Phi$  is real-analytic in  $\mathbb{B}^{n+1}$ , so that in particular  $\Phi$  is continuous whenever  $\phi$  is continuous.

*Proof.* Recall that Euclidean balls  $\mathbb{B}^{n+1}(\underline{\zeta}, r)$  and spheres  $\mathbb{S}^n(\underline{\zeta}, r)$  are mapped to such balls and spheres (possibly half spaces and hyperplanes or complements of a closed ball union  $\infty$ ) under any conformal automorphism  $g \in G$ .

Thus given a spherical ball  $B_{\mathbb{S}^n}(\underline{\zeta}, \delta) \subset \mathbb{S}^n$ ,  $0 < \delta < \pi$  and  $\mathbf{w} \in \mathbb{B}^{n+1}$  there are two alternative ways of describing the size of  $B_{\mathbb{S}^n}(\underline{\zeta}, \delta)$  viewed from  $\mathbf{w}$ . Either we can use the visual Poincaré radius from  $\mathbf{w}$ , i.e. the spherical radius of the ball  $g_{-\mathbf{w}}(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  in  $\mathbb{S}^n$  or we can use the  $\mathbf{w}$ -harmonic measure  $\eta_{\mathbf{w}}(B_{\mathbb{S}^n}(\underline{\zeta}, \delta)) = \eta_{\mathbf{0}}(g_{-\mathbf{w}}(B_{\mathbb{S}^n}(\underline{\zeta}, \delta)))$ .

Given  $B_{\mathbb{S}^n}(\underline{\zeta}, \delta)$  we denote by  $W(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  the set  $B_{\mathbb{S}^n}(\underline{\zeta}, \delta)$  itself union the open subset of points  $\mathbf{w} \in \mathbb{B}^{n+1}$  for which the visual Poincaré radius from  $\mathbf{w}$  exceeds  $\pi/4$ . Similarly we denote by  $U(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  the set  $B_{\mathbb{S}^n}(\underline{\zeta}, \delta)$  itself union the open subset of points  $\mathbf{w} \in \mathbb{B}^{n+1}$  for which  $\eta_{\mathbf{w}}(B_{\mathbb{S}^n}(\underline{\zeta}, \delta)) > 2/3$ . Then  $U(B_{\mathbb{S}^n}(\underline{\zeta}, \delta)) \subset W(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  and both sets are neighborhoods of  $\underline{\zeta}$  in  $\overline{\mathbb{B}}^{n+1} = \mathbb{B}^{n+1} \cup \mathbb{S}^n$ . In fact for  $\underline{\zeta} = \mathbf{e}_1$  and  $\delta = \pi/4$  the set  $W(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  equals the intersection of  $\overline{\mathbb{B}}^{n+1}$  with the open ball  $\mathbb{B}^{n+1}(\sqrt{2}\mathbf{e}_1, 1)$  and for  $\delta = 2\pi/3$  the open set  $U(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$  is the complement  $\overline{\mathbb{B}}^{n+1} \setminus \overline{\mathbb{B}}^{n+1}(-2\mathbf{e}_1, \sqrt{3})$ . Clearly any of the families of sets  $U(B_{\mathbb{S}^n}(\underline{\zeta}, \delta)), W(B_{\mathbb{S}^n}(\underline{\zeta}, \delta))$ ,  $0 < \delta < \pi$  forms fundamental systems of neighbourhoods of  $\underline{\zeta}$  in  $\overline{\mathbb{B}}^{n+1}$ . Suppose  $\phi$  is continuous at  $\underline{\zeta}_0$  and let  $0 < \epsilon < \pi$  be given. Choose  $0 < \delta < \pi$  such that

$$\phi(B_{\mathbb{S}^n}(\underline{\zeta}_0, \delta)) \subset B_{\mathbb{S}^n}(\phi(\underline{\zeta}_0, \epsilon)).$$

Then for any  $\mathbf{w} \in U(B_{\mathbb{S}^n}(\underline{\zeta}_0, \delta))$  and any  $\mathbf{z} \in \partial W(B_{\mathbb{S}^n}(\phi(\underline{\zeta}_0, \epsilon))) \cap \mathbb{B}^{n+1}$  the vector  $V_{\phi_*\eta_{\mathbf{w}}}(\mathbf{z})$  points into  $W(B_{\mathbb{S}^n}(\phi(\underline{\zeta}_0, \epsilon)))$ . Hence  $\Phi(\mathbf{w})$  the unique zero of  $V_{\phi_*\eta_{\mathbf{w}}}$  belongs to  $W(B_{\mathbb{S}^n}(\phi(\underline{\zeta}_0, \epsilon)))$ . This proves continuity at  $\underline{\zeta}_0$ . □

**Lemma 9.** *Let  $\phi \in \mathcal{E}(\mathbb{S}^n)$  and  $E(\phi) = \Phi$  be as above. Then  $\Phi$  is real-analytic in  $\mathbb{B}^{n+1}$ .*

*Proof.* Towards real-analyticity of  $\Phi$  recall that  $\Phi(\mathbf{z})$  is the unique zero of the vector field

$$\begin{aligned} V_{\phi_*(\eta_{\mathbf{z}})}(\mathbf{w}) &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} \underline{\zeta} (g_{-\mathbf{w}} \circ \phi)_* \eta_{\mathbf{z}}(\underline{\zeta}) \\ &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\phi(\underline{\zeta})) (g_{-\mathbf{z}})_* \eta_{\mathbf{0}}(\underline{\zeta}) \\ &= \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\phi(\underline{\zeta})) \left( \frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \underline{\zeta}|^2} \right)^n d\eta_{\mathbf{0}}(\underline{\zeta}). \end{aligned}$$

Thus  $\forall \mathbf{z} \in \mathbb{B}^{n+1}$  the value  $\mathbf{w} = \Phi(\mathbf{z})$  is the unique point  $\mathbf{w} \in \mathbb{B}^{n+1}$  such that:

$$F(\mathbf{z}, \mathbf{w}) = \frac{2V_{\phi_*(\eta_{\mathbf{z}})}(\mathbf{w})}{1 - |\mathbf{w}|^2} = \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\phi(\underline{\zeta})) \left( \frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \underline{\zeta}|^2} \right)^n d\eta_{\mathbf{0}}(\underline{\zeta}) = \mathbf{0}.$$

Clearly  $F$  is a real-analytical function of  $(\mathbf{z}, \mathbf{w}) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$ . Thus by the implicit function theorem we need only show that for any pair  $(\mathbf{z}, \mathbf{w}) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$  with  $F(\mathbf{z}, \mathbf{w}) = \mathbf{0}$  the  $\mathbf{w}$  partial derivatives matrix  $\mathbf{J}_{\mathbf{w}}F = \frac{\partial F}{\partial \mathbf{w}}$  evaluated at  $(\mathbf{z}, \mathbf{w})$  is non-singular. By conformal naturality we can suppose  $\mathbf{z} = \mathbf{w} = \Phi(\mathbf{z}) = \mathbf{0}$ . A straight forward computation analogous to the one leading to (6) yields that  $\mathbf{J}_{\mathbf{w}}F$  evaluated at  $(\mathbf{0}, \mathbf{0})$  and applied to the vector  $\underline{\epsilon}$  is given by the formula:

$$\mathbf{J}_{\mathbf{w}}F(\underline{\epsilon}) = -2 \int_{\mathbb{S}^n} (\underline{\epsilon} - \langle \underline{\epsilon}, \phi(\underline{\zeta}) \rangle \phi(\underline{\zeta})) d\eta_{\mathbf{0}}(\underline{\zeta}).$$

Similarly as for  $\text{Jac}_V$  this shows that  $\mathbf{J}_{\mathbf{w}}F$  is non singular at  $(\mathbf{0}, \mathbf{0})$ . So  $\Phi$  is real-analytic by the implicit function theorem and

$$(11) \quad \text{Jac}_{\Phi}(\mathbf{0}) = -(\mathbf{J}_{\mathbf{w}}F)^{-1} \circ \mathbf{J}_{\mathbf{z}}F$$

where both partial derivative matrices are evaluated at  $(\mathbf{0}, \mathbf{0})$  and

$$\mathbf{J}_{\mathbf{z}}F = \int_{\mathbb{S}^n} \phi(\underline{\zeta}) \times \underline{\zeta} d\eta_{\mathbf{0}}(\underline{\zeta}),$$

and where  $\phi(\underline{\zeta}) \times \underline{\zeta}$  is the matrix valued mapping

$$A_{i,j}(\underline{\zeta}) = \phi_i(\underline{\zeta}) \cdot \zeta_j.$$

□

#### 4. PROPERTIES OF THE D-E EXTENSION OF RATIONAL MAPS

The question that naturally arises is: For  $f$  a rational map on the Riemann sphere. What are the geometric and dynamical properties of the D-E extension  $E(f)$ ? How many of the properties of  $f$  are inherited by  $E(f)$ . By elementary topology  $E(f)$  is a proper map, that is the preimage of any compact set is compact. And moreover for any point  $\mathbf{w} \in \overline{\mathbb{B}^{n+1}}$  the pre image  $E(f)^{-1}(\mathbf{w})$  is a real analytic set.

Question 1: Is  $E(f)$  a discrete map?

Question 2: Is  $E(f)$  an open map?

Question 3: Is  $E(f)$  a map of the same degree as  $f$ ?



Question 4: Is the Julia set of  $E(f)$  (the set of points  $\mathbf{x}$  for which the family of iterates does not form an equicontinuous family on any neighbourhood of  $\mathbf{x}$ ) equal to the convex hull of the Julia set for  $\hat{f}$ ?

In certain elementary cases at least the immediate answer to the above questions are yes, but not completely satisfactory.

For the following discussion we shall identify  $\mathbb{C}$  with the coordinate plane in  $\mathbb{R}^3$ ,  $\{\mathbf{x} = (x_1, x_2, x_3) \mid x_3 = 0\}$  and write  $z = x + iy$  for the point  $(x, y, 0)$ . In particular we shall identify the complex unit disk  $\mathbb{D}$  with the disc  $\{\mathbf{x} \in \mathbb{R}^3 \mid |z|^2 = x_1^2 + x_2^2 < 1, x_3 = 0\}$  and the unit circle  $\mathbb{S}^1$  with the circle  $\{\mathbf{x} \in \mathbb{R}^3 \mid |z|^2 = 1, x_3 = 0\}$ . Then stereographic projection  $S$  of  $\overline{\mathbb{C}}$  on to  $\mathbb{S}^2$  from the north pole  $N = \mathbf{e}_3 \in \mathbb{R}^3$  is the map

$$z \mapsto S(z) = \left( \frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right) = \frac{1}{1 + |z|^2} (2x, 2y, 1 - |z|^2).$$

For  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  a holomorphic map we shall write  $\hat{f}$  for its conjugate by  $S$ , i.e.:

$$\hat{f}(S(z)) = S(f(z)).$$

In the following we shall discuss finite Blaschke products

$$f(z) = \sigma \prod_{j=1}^d \frac{z + a_j}{1 + \overline{a_j}z}, \quad |\sigma| = 1, \quad a_j \in \mathbb{D}$$

**Proposition 10.** *For  $f$  a finite Blaschke product the D-E extension  $E(\hat{f})$  maps  $\mathbb{D}$  onto  $\mathbb{D}$ , preserves the upper and lower hemispheres,  $\mathbb{S}^2_+, \mathbb{S}^2_-$  and further more on  $\mathbb{D}$  we have  $\partial E(\hat{f})/\partial x_3 = g(z)\mathbf{e}_3$  for some positive real analytical function  $g : \mathbb{D} \rightarrow \mathbb{R}_+$ .*

*If moreover  $f(z) = z^d$  (i.e.  $a_j = 0$  for all  $j$ ), then  $E(\hat{f})(z) = z^d \cdot h(|z|^2)$  for some real analytical function  $h$  with  $h(r) \rightarrow 1$  as  $r \nearrow 1$ .*

*Proof.* The reflection  $c(x_1, x_2, x_3) = (x_1, x_2, -x_3)$  is the Poincaré extension of  $\hat{\tau}$ , where  $\tau(z) = 1/\bar{z}$  denotes the reflection in  $\mathbb{S}^1$ . Then  $c \circ \hat{f} \circ c = \hat{f}$ . Write  $\Phi = E(\hat{f})$  for the D-E extension of  $\hat{f}$ . Then by conformal naturality of the D-E extension

$$(12) \quad c \circ \Phi = \Phi \circ c.$$

Hence  $\Phi(\mathbb{D}) \subseteq \mathbb{D}$ , since  $c|_{\mathbb{D}} : \mathbb{D} \rightarrow \mathbb{D}$  is the identity. Moreover if  $\Phi(\mathbb{D}) \neq \mathbb{D}$ , then a simple homotopy argument would imply that the restriction  $\hat{f}|_{\mathbb{S}^1} = \Phi|_{\mathbb{S}^1}$  is homotopic to a constant map. Thus  $\Phi(\mathbb{D}) = \mathbb{D}$ .

To prove that  $\Phi$  preserves the upper and hence the lower hemisphere it suffices to prove that for any  $\mathbf{x} \in \mathbb{S}^2_+$  and any  $w \in \mathbb{D}$ :  $\mathbf{e}_3 \cdot V_{\hat{f}_* \eta_{\mathbf{x}}}(w) > 0$ . Furthermore by conformal naturality it suffices to consider the case  $\mathbf{x} = t\mathbf{e}_3$  with  $0 < t < 1$  and  $w = 0$ . Before we start computing let us note that since  $c_* \eta_{\mathbf{0}} = \eta_{\mathbf{0}}$  we have for any measurable function  $\phi : \mathbb{S}^n \rightarrow \mathbb{R}(\mathbb{C})$

$$\int_{\mathbb{S}^n} \phi(\underline{\zeta}) d\eta_{\mathbf{0}} = \int_{\mathbb{S}^n_+} (\phi(\underline{\zeta}) + \phi(c(\underline{\zeta}))) d\eta_{\mathbf{0}}(\underline{\zeta}) = \int_{\mathbb{S}^n_-} (\phi(\underline{\zeta}) + \phi(c(\underline{\zeta}))) d\eta_{\mathbf{0}}(\underline{\zeta}).$$

Applying this to  $V_{\widehat{f}_* \eta_{\mathbf{x}}}(0)$  we obtain

$$\begin{aligned} \mathbf{e}_3 \cdot V_{\widehat{f}_* \eta_{\mathbf{x}}}(0) &= \int_{\mathbb{S}^2} \mathbf{e}_3 \cdot \widehat{f}(\underline{\zeta}) \left( \frac{1 - |\mathbf{x}|^2}{|\mathbf{x} - \underline{\zeta}|^2} \right)^2 d\eta_{\mathbf{0}}(\underline{\zeta}) \\ &= \int_{\mathbb{S}^2_+} \left( \widehat{f}_3(\underline{\zeta}) \left( \frac{1 - |\mathbf{x}|^2}{|\mathbf{x} - \underline{\zeta}|^2} \right)^2 + \widehat{f}_3(c(\underline{\zeta})) \left( \frac{1 - |\mathbf{x}|^2}{|\mathbf{x} - c(\underline{\zeta})|^2} \right)^2 \right) d\eta_{\mathbf{0}}(\underline{\zeta}) \\ &= \int_{\mathbb{S}^2_+} \widehat{f}_3(\underline{\zeta}) \left( \left( \frac{1 - |\mathbf{x}|^2}{|\mathbf{x} - \underline{\zeta}|^2} \right)^2 - \left( \frac{1 - |\mathbf{x}|^2}{|\mathbf{x} - c(\underline{\zeta})|^2} \right)^2 \right) d\eta_{\mathbf{0}}(\underline{\zeta}) > 0, \end{aligned}$$

since  $|\mathbf{x} - \underline{\zeta}| < |\mathbf{x} - c(\underline{\zeta})|$  and  $\widehat{f}_3$  is positive on  $\mathbb{S}^2_+$ .

To compute the partial derivative vector  $\partial\Phi/\partial x_3(z)$  for  $z \in \mathbb{D}$  we equate the Jacobians of the two sides of (12) and obtain

$$\frac{\partial\Phi_1}{\partial x_3}(z) = \frac{\partial\Phi_2}{\partial x_3}(z) = \frac{\partial\Phi_3}{\partial x_1}(z) = \frac{\partial\Phi_3}{\partial x_2}(z) = 0.$$

Thus  $\partial\Phi/\partial x_3(z) = \partial\Phi_3/\partial x_3(z)\mathbf{e}_3 = g(z)\mathbf{e}_3$ . To complete the first set of statements we just need to show that  $g$  is a positive function. By conformal naturality it suffices to consider the case  $\Phi(0) = 0$ . Furthermore in order to simplify notation let us for  $\phi : \mathbb{S}^n \rightarrow \mathbb{R}(\mathbb{C})$  a measurable function write

$$M(\phi) := \int_{\mathbb{S}^n} \phi(\underline{\zeta}) d\eta_{\mathbf{0}}(\underline{\zeta}).$$

Then an elementary calculation, using (11) yields

$$g(0) = \frac{\partial\Phi_3}{\partial x_3}(0) = \frac{M(\widehat{f}_3 \cdot \zeta_3)}{2M(1 - \widehat{f}_3^2)} > 0,$$

since  $\widehat{f}_3 \cdot \zeta_3, (1 - \widehat{f}_3^2) \geq 0$  with equality for  $\zeta_3 = 0$  only in the first and for  $f(z)$  equal to 0 or  $\infty$  in the second.

In the special case  $f(z) = z^d$  we have  $f(e^{i\theta}z) = e^{id\theta}f(z)$ . and thus by conformal naturality  $\Phi(e^{i\theta}z) = e^{id\theta}\Phi(z)$ . Hence  $\Phi(0) = 0$  and for  $z \neq 0$ :

$$\Phi(z) = \Phi(|z|) \frac{z^d}{|z|^d}.$$

Moreover  $f$  commutes with complex conjugation, which translates to  $\widehat{f}$  commutes with the reflection  $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$ . As above this implies that  $\Phi([-1, 1] \subseteq [-1, 1])$ , so that  $\Phi$  is a real analytic real function on the reals. Expanding the real-analytic function in a power series in  $z, \bar{z}$  on a neighbourhood of 0 and noting that  $z\bar{z} = |z|^2$  we obtain:

$$\Phi(z) = \frac{z^d}{|z|^d} \sum_{m=0}^{\infty} b_m |z|^m = \sum_{n,k=0}^{\infty} a_{n,k} z^n \bar{z}^k.$$

By the uniqueness theorem for power series this implies that  $a_{n,k} = 0$  for  $n - k \neq d$ . Thus we are left with

$$\Phi(z) = \sum_{k=0}^{\infty} a_{k+d,k} z^{k+d} \bar{z}^k = z^d \sum_{k=0}^{\infty} a_{k+d,k} |z|^{2k}.$$

□

Write  $M_t(z) = tz$  for  $0 < t$  so that  $M_t$  is a homothety. Then the conformal automorphism  $h_t = E(\widehat{M}_t) = g_{\mathbf{w}}$  with  $\mathbf{w} = \mathbf{w}(t) = \frac{t-1}{t+1} \mathbf{e}_3$  maps  $\mathbb{D}$  conformally onto the geodesic disk  $\mathbb{D}_t$  in  $\mathbb{B}^3$  with boundary the circle  $\widehat{M}_t(\mathbb{S}^1)$ .

**Corollary 11.** *For  $f(z) = z^d$  the D-E extension  $E(\widehat{f})$  maps  $\mathbb{D}_t$  onto  $\mathbb{D}_{t^d}$  by a degree  $d$  ramified covering and the interval  $[0, \mathbf{e}_3[$  onto itself by an increasing diffeomorphism.*

**Conjecture 1.** *For all finite Blaschke products  $f$  we have  $f = E(\widehat{f})$  on  $\mathbb{D}$ .*

By conformal naturality of the D-E extension this conjecture is equivalent to the seemingly simpler conjecture:

**Conjecture 2.** *For all finite Blaschke products  $f$  with  $f(0) = 0$  we have  $E(\widehat{f})(0) = 0$ .*

If true the following stronger conjecture would yield almost complete topological understanding of  $E(\widehat{f})$  for any finite Blaschke product  $f$ :

**Conjecture 3.** *For all finite Blaschke products  $f$  with  $f(0) = 0$  the D-E extension  $E(\widehat{f})$  maps the interval  $[0, \mathbf{e}_3[$  diffeomorphically and increasingly onto itself.*

Clearly the last conjecture implies the two previous ones. Moreover by conformal naturality of the D-E extension, it would imply that for each  $z \in \mathbb{D}$  the unique hyperbolic geodesic through  $z$  and orthogonal to  $\mathbb{D}$  would be mapped diffeomorphically onto the unique such geodesic through  $f(z)$ . And thus the dynamics of  $E(\widehat{f})$  would be conjugate to a skew product on  $\mathbb{D} \times [-1, 1]$ . Which would be completely analogous to the case of Fuchsian groups.

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